

Ion trap analogue of particle creation in black holes and cosmology

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We consider the transversal modes of ions in a linear radio frequency (rf) trap where we control the time dependent axial confinement to show that we can excite quanta of motion via a two-mode squeezing process. This effect is analogous to phenomena predicted to occur during the evaporation of black holes and cosmological particle creation, in general out of reach for experimental investigation. As substantial advantage of this proposal in comparison to previous ones we propose to exploit radial and axial modes simultaneously to permit experimental access of these effects based on state-of-the-art technology. In addition, we propose to create and explore entanglement, starting with two ions, and relate the results to fundamental aspects of the entropy of black holes.

I. INTRODUCTION

It is a fundamental prediction of quantum field theory that extreme conditions, such as non-adiabatic dynamics, can create pairs of particles out of the quantum vacuum. Examples are Hawking radiation, i.e., black hole evaporation, and cosmological particle creation [1]. To provide an intuitive picture of such an effect, let us imagine two pendula coupled by a spring. The classical ground states with and without spring remain identical, however, the ground states of the quantum version differ in a fundamental way. Without the spring, we describe the system by a product of the individual ground states, while the two coupled pendula require entanglement of the non-separable state, see [2]. Now, we remove the spring instantaneously such that the system has no time to evolve to react, we end up with two pendula which are not in their individual ground states, i.e., excited. The entanglement of the state corresponds to the correlation between the two pendula, e.g., if pendulum one was in the first excited state, then the second one has to match the excitation – while the total state of the system remains a pure state.. This entanglement also implies that if we consider one pendulum only, by tracing over the degrees of freedom of the second one, the effective state of pendulum one will be indistinguishable from a thermal (i.e., mixed) state.

In quantum field theory, this instantaneous or non-adiabatic removal of the spring is predicted to be caused by extreme circumstances, such as during the inflationary part of the expansion of the universe or in vicinity of a black hole, when wave-packets get torn apart. In the latter case, the entanglement between the two “pendula” (one inside and the other outside the horizon) explains the thermal character of Hawking radiation. Here, we propose an experimentally realizable analogue of this effects based on trapped ions. The radial modes of the two or more ions represent the two quantum pendula

while the spring is analogous to their Coulomb interaction within the axial trapping potential. We define the amplitude and the evolution in time of the latter by applying potentials to additional electrodes, controlling the axial motion of the ions and their mutual distance, respectively. Due to the unique control and accurate detection of the electronic and motional degrees of freedom, trapped ions are very good candidates for investigating these quantum effects, see also [3]. Further examples for the simulation of relativistic effects in ion traps can be found in [4–6].

II. EXCITATION OF PHONONS

We investigate a system of N ions of the identical species in a harmonic trapping potential characterized by a constant radial secular frequency ω_{rad}^2 , provided by time-averaging the rf-potential. In axial direction, we specify the time-dependent confinement by $\omega_{\text{ax}}^2(t)$. This system is a generalization of the one-dimensional approach treated in [3], where only the motion along the axial direction has been investigated. Here we assume that the radial confinement is always stronger than the axial one, i.e., $\omega_{\text{rad}}^2 > \omega_{\text{ax}}^2(t)$. The classical equation of motion of the k -th ion with coordinate \mathbf{r}_k reads then

$$\ddot{\mathbf{r}}_k + \begin{pmatrix} \omega_{\text{ax}}^2(t) & 0 & 0 \\ 0 & \omega_{\text{rad}}^2 & 0 \\ 0 & 0 & \omega_{\text{rad}}^2 \end{pmatrix} \cdot \mathbf{r}_k = \gamma \sum_{l \neq k}^N \frac{\mathbf{r}_k - \mathbf{r}_l}{|\mathbf{r}_k - \mathbf{r}_l|^3}, \quad (1)$$

where the constant γ encodes the strength of the Coulomb repulsion between the ions. In the following, we focus on solutions of (1) starting in the static equilibrium positions $\mathbf{r}_k(t_{\text{in}}) := \mathbf{r}_k^{\text{eq}} := (x_k^{\text{eq}}, 0, 0)^T$. The solutions of (1) can be written as $\mathbf{r}_k(t) = b(t)\mathbf{r}_k^{\text{eq}}$, where the scale parameter $b(t)$ fulfills

$$\ddot{b}(t) + \omega_{\text{ax}}^2(t)b(t) = \frac{(\omega_{\text{ax}}^{\text{in}})^2}{b^2(t)}. \quad (2)$$

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The boundary conditions are $b(t_{\text{in}}) = 1$ and $\dot{b}(t_{\text{in}}) = 0$. This means the classical solution is fully determined as a time dependent rescaling of the initial equilibrium positions.

However, the ions are quantum particles described by a wave function of a certain width, individual measurements of their positions have to deviate from and fluctuate around their classically predictable positions, revealing quantum fluctuations. Their position operator can be written as $\hat{\mathbf{r}}_k(t) = b(t)\mathbf{r}^{\text{eq}} + \delta\hat{\mathbf{r}}_k$ and in a semiclassical approximation, we assume that the deviations $\delta\hat{\mathbf{r}}_k$ remain small (because the mass of the ions being large corresponding to a narrow width of their ground state wave function).

Linearization and diagonalization of (1) yields then the Heisenberg equation of motion for the normal modes (phonons). While the axial phonons have been discussed in [3] we focus here on the radial phonons satisfying

$$\left(\frac{d^2}{dt^2} + \Omega_\kappa^2(t)\right) \delta\hat{y}_\kappa = 0. \quad (3)$$

Every radial normal mode $\delta\hat{y}_\kappa$ can be associated to one individual harmonic oscillator with time dependent normal mode frequency

$$\Omega_\kappa^2(t) = \omega_{\text{rad}}^2 - \frac{\omega_\kappa^2}{b^3(t)} \quad (4)$$

where $\omega_\kappa^2 \geq 0$ is the κ -th eigenvalue of the matrix

$$M_{kl} = \delta_{kl} \sum_{j \neq k}^N \frac{\gamma}{|x_k^{\text{eq}} - x_j^{\text{eq}}|^3} - \frac{\gamma(1 - \delta_{kl})}{|x_k^{\text{eq}} - x_l^{\text{eq}}|^3}. \quad (5)$$

Especially for the center of mass mode we have $\omega_0 = 0$ and for the rocking mode $\omega_1 = \omega_{\text{ax}}^{\text{in}}$.

In the following we show how the time-dependence of the normal mode frequencies $\Omega_\kappa(t)$ can lead to the excitation of phonons. At the initial instant t_{in} we express the position operator of each normal mode in terms of the harmonic oscillator ladder operators as

$$\delta\hat{y}_\kappa(t_{\text{in}}) = \frac{1}{\sqrt{2\Omega_\kappa(t_{\text{in}})}} \hat{a}_\kappa^{\text{in}} + \text{h.c.} \quad (6)$$

For another given instant $t_{\text{out}} > t_{\text{in}}$, the operator evolves under the Heisenberg equation (3) into

$$\delta\hat{y}_\kappa(t_{\text{out}}) = \frac{1}{\sqrt{2\Omega_\kappa(t_{\text{out}})}} \hat{a}_\kappa^{\text{out}} + \text{h.c.}, \quad (7)$$

where the final creation/annihilation operators $\hat{a}_\kappa^{\text{out}}/\hat{a}_\kappa^{\text{out}\dagger}$ are linked to the initial ones via the Bogoliubov transformation

$$\hat{a}_\kappa^{\text{out}} = \alpha_\kappa^* \hat{a}_\kappa^{\text{in}} - \beta_\kappa^* \hat{a}_\kappa^{\text{in}\dagger}. \quad (8)$$

with the (complex) Bogoliubov coefficients α_κ and β_κ . For the initial ground state $|\Psi(t_{\text{in}})\rangle = |0\rangle$ in the κ -th radial mode, the mean number of created phonons is given by

$$\langle \hat{n}_\kappa^{\text{out}} \rangle = \langle \Psi(t_{\text{in}}) | \hat{a}_\kappa^{\text{out}\dagger} \hat{a}_\kappa^{\text{out}} | \Psi(t_{\text{in}}) \rangle = |\beta_\kappa|^2. \quad (9)$$

Hence, phonon creation takes place depending on the temporal evolution of $\Omega_\kappa(t)$ from t_{in} to t_{out} , if $|\beta_\kappa| > 0$. Or in other words: The classical motion along the x -axis induces the creation of phonons in the radial direction.

The generators of the Bogoliubov transformation (8) are Squeezing Operators. Therefore the time evolution of the initial ground state $|\Psi(t_{\text{in}})\rangle$ is given by

$$\begin{aligned} |\Psi(t_{\text{out}})\rangle &= \hat{S}_\xi |0\rangle \exp \left\{ \frac{1}{2} \sum_\kappa \xi_\kappa (\hat{a}_\kappa^{\text{in}\dagger})^2 - \text{h.c.} \right\} |0\rangle \\ &= |0\rangle + \frac{1}{\sqrt{2}} \sum_\kappa \xi_\kappa |2_\kappa\rangle + \mathcal{O}(\xi_\kappa^2). \end{aligned} \quad (10)$$

where the Squeezing parameter ξ_κ is linked to the Bogoliubov coefficients via $|\beta_\kappa| = \sinh |\xi_\kappa|$ and $\arg \xi_\kappa = -(\arg \alpha_\kappa + \arg \beta_\kappa)$. Formula (10) features the characteristics of a squeezing operation, the creation of particles (here phonons) in pairs.

III. EXCITATION MODELS FOR TWO IONS

In the following we focus on the case of $N = 2$ ions and investigate the phonon creation induced by different axial motions of the ions. Firstly by a collision between the ions described by a scale functions $b_{\text{col}}(t)$ and secondly by an expansion of the ions corresponding to a scale function $b_{\text{ex}}(t)$. The time dependence of the axial confinement necessary to generate a given scale function $b(t)$ can be deduced from (2) to

$$\omega_{\text{ax}}(t) = \sqrt{\frac{(\omega_{\text{ax}}^{\text{in}})^2}{b^3(t)} - \frac{\ddot{b}(t)}{b(t)}}. \quad (11)$$

We focus here on trajectories where $\omega_{\text{ax}}(t) \in \mathbb{R}$. However, there exist also trajectories $b(t)$ that can only be realized for temporarily negative ω_{ax}^2 , that means for temporarily repulsive trapping potentials.

The scale function is linked to the (classical) mutual distance of the ions via

$$\Delta x(t) = x_1(t) - x_2(t) = b(t) \Delta x^{\text{eq}}. \quad (12)$$

In the radial direction we have the two phonon modes

$$\delta\hat{y}_\pm = \frac{1}{\sqrt{2}} (\delta\hat{y}_1 \pm \delta\hat{y}_2). \quad (13)$$

This is the center of mass mode $\delta\hat{y}_+$ with the frequency $\Omega_+^2 = \omega_{\text{rad}}^2$ and the rocking mode $\delta\hat{y}_-$ with the frequency $\Omega_-^2(t) = \omega_{\text{rad}}^2 - (\omega_{\text{ax}}^{\text{in}})^2/b(t)^3$. With (4) the equation of motion for the rocking mode phonons is

$$\left(\frac{d^2}{dt^2} + \Omega_-^2(t)\right) \delta\hat{y}_- = 0. \quad (14)$$

A. Collision model

We consider now a special scale function

$$b_{\text{col}}(t) = \left(1 + \frac{\Delta\Omega_{\text{col}}^2}{(\omega_{\text{ax}}^{\text{in}})^2} \frac{1}{\cosh^2(\omega_{\text{col}} t)} \right)^{-\frac{1}{3}} \quad (15)$$

that parametrizes a collision between the ions. Starting at $t_{\text{in}} \rightarrow -\infty$ in the equilibrium position with $b_{\text{col}}(t_{\text{in}}) = 1$, the ions approach each other, reach for $t = 0$ a minimal axial distance Δx_{min} at the turning point and finally return to their initial positions for $t_{\text{out}} \rightarrow +\infty$ with $b(t_{\text{out}}) = 1$. The parameter $\Delta\Omega_{\text{col}}^2$ describes the change in the rocking mode frequency Ω_-^2 from t_{in} to $t = 0$ and determines the minimal distance of the ions. The parameter ω_{col} determines the characteristic time scale of the collision. Eq. (14) can be solved for $b_{\text{col}}(t)$ in terms of hypergeometric functions whose asymptotic behaviour is known for $t \rightarrow \pm\infty$. As shown in appendix A this yields the Bogoliubov coefficient

$$|\beta_-^{\text{col}}|^2 = \left| \frac{\cosh\left(\frac{\pi}{2} \sqrt{\frac{4\Delta\Omega_{\text{col}}^2}{\omega_{\text{col}}^2} - 1}\right)}{\sinh\left(\frac{\pi\Omega_{\text{in}}}{\omega_{\text{col}}}\right)} \right|^2, \quad (16)$$

where $\Omega_{\text{in}} = \Omega_-(t_{\text{in}})$. Here we focus on a regime of moderate and slow collisions, where $\omega_{\text{col}} \ll \Delta\Omega_{\text{col}} < \Omega_{\text{in}}$. Especially, this implies that the system never reaches critical points with $\Omega_- = 0$, where the classical radial motion becomes unstable and the linear chain features a phase transition into a two-dimensional zig-zag structure [7]. Under these assumptions, Eq. (16) can be approximated as

$$|\beta_-^{\text{col}}|^2 \approx \exp\left[-2\pi \frac{(\Omega_{\text{in}} - \Delta\Omega_{\text{col}})}{\omega_{\text{col}}}\right]. \quad (17)$$

That means that particle creation becomes only important if $\Omega_{\text{in}} - \Delta\Omega_{\text{col}}$ is chosen sufficiently small

$$\Omega_{\text{in}} - \Delta\Omega_{\text{col}} = \mathcal{O}(\omega_{\text{col}}) \quad (18)$$

and is exponentially suppressed for $\Omega_{\text{in}} - \Delta\Omega_{\text{col}} \gg \omega_{\text{col}}$. In fact, this statement is valid for generic scale functions, as long as the collision fulfills the given assumptions. Performing a WKB approximation in appendix B we show that the mean number of phonons is mainly dominated by the relation between two parameters: the normal mode frequency $\Omega_-^2(0)$ and its curvature $\frac{d^2}{dt^2}\Omega_-^2(0)$, both evaluated at the turning point, which occurs at $t = 0$. As $\Omega_-(t)^2$ is linked to the ion trajectories via (4) and (1) it is equivalent to state that the mean number of phonons after the collision is mainly dominated by the two parameters

$$p_1 := \left(\frac{\Delta x^{\text{eq}}}{\Delta x^{\text{min}}} \right)^3 \quad (19)$$

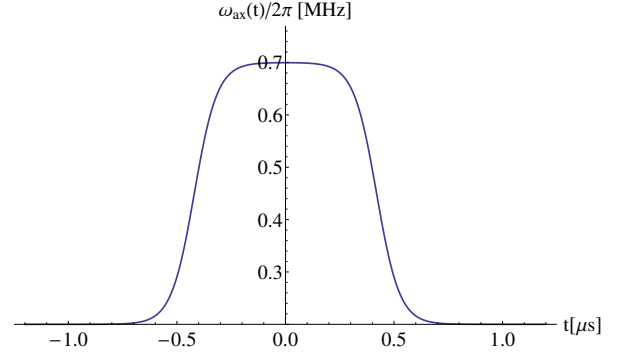


FIG. 1. Example of a time dependent axial confinement characterized by $\omega_{\text{ax}}(t)$ that leads to the classical motion illustrated in Fig. 2.

and

$$p_2 := \left(\frac{\omega_{\text{ax}}(t=0)}{\omega_{\text{ax}}^{\text{in}}} \right)^2, \quad (20)$$

where $\omega_{\text{ax}}(t=0)$ describes the axial confinement at the instance when the ions reach their turning point. For example a model collision with trajectory $b_{\text{col}}(t)$ yielding the two related values p_1 and p_2 is obtained by choosing

$$\Delta\Omega_{\text{col}}^2(p_1, p_2) = (\omega_{\text{ax}}^{\text{in}})^2 (p_1 - 1) \quad (21)$$

and

$$\omega_{\text{col}}^2(p_1, p_2) = (\omega_{\text{ax}}^{\text{in}})^2 \frac{3p_1(p_1 - p_2)}{2(p_1 - 1)}. \quad (22)$$

We can take advantage of this to obtain approximations for the Bogoliubov coefficients β_- of moderate and slow collision with trajectories qualitatively similar to (16). Such a collision with given parameters p_1 and p_2 will lead to similar phonon excitations as model collisions (16) having identical parameters. Therefore the Bogoliubov coefficient $|\beta_-|^2$ can be approximated as

$$|\beta_-|^2 \approx |\beta_-^{\text{col}}(p_1, p_2)|^2, \quad (23)$$

where $\beta_-^{\text{col}}(p_1, p_2)$ denotes β_-^{col} from (16) with the substitutions (21) and (22).

Let us exploit these results to propose a realistic implementation: a collision of two $^{25}\text{Mg}^+$ ions, trapped in a radial potential with frequency $\omega_{\text{rad}} = 2\pi \cdot 3.5$ MHz and an initial axial potential with frequency $\omega_{\text{ax}}^{\text{in}} = 2\pi \cdot 0.2$ MHz. The initial equilibrium distance is $\Delta x^{\text{eq}} \approx 19.1 \mu\text{m}$. As an example we consider the axial confinement presented in Fig. 1, where we increase $\omega_{\text{ax}}^{\text{in}}$ in approximately $0.5 \mu\text{s}$ to $\omega_{\text{ax}}^{\text{max}} = 2\pi \cdot 0.7$ MHz, keep it constant for around $0.5 \mu\text{s}$ and return to $\omega_{\text{ax}}^{\text{in}}$. In Fig. 2 the resulting ion trajectory is illustrated. The exact Bogoliubov coefficient can be evaluated either numerically to $|\beta_-|^2 \approx 0.18$ or approximately based on (23) to $|\beta_-|^2 \approx 0.2$. A more extensive comparison between the approximation (23) and

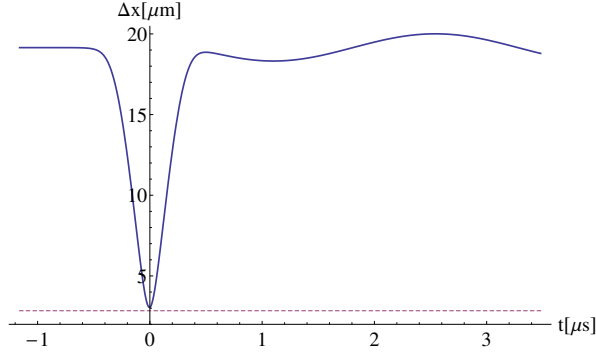


FIG. 2. Numerically calculated trajectory Δx for two ions confined in the axial potential of its characteristic $\omega_{\text{ax}}(t)$ presented in Fig. 1. After the collision the ions oscillate relative to their initial equilibrium positions. The red dashed line shows the critical distance at which $\Omega_- = 0$ where the ion chain becomes unstable.

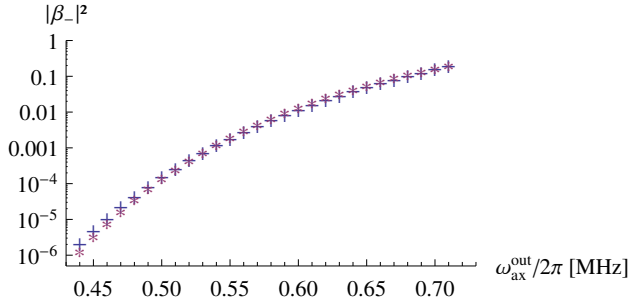


FIG. 3. Bogoliubov coefficient β_- obtained (+) numerically or (*) based on Eq. (23) for collisions induced by an axial potential as presented in Fig. 1, where the peak confinement $\omega_{\text{ax}}^{\text{max}}$ is varied.

the exact numerical results can be carried out by calculating the Bogoliubov coefficients for different final confinement $\omega_{\text{ax}}^{\text{max}}$ by both methods. The result is illustrated in Fig. 3 and indicates that we achieve good agreement over several orders of magnitude.

Furthermore this realistic results permit predicting that the mean phonon numbers created in the radial mode can be five times larger than the residual thermal excitation of $n_{\text{th}} \approx 0.05$, achievable by current cooling techniques. In addition, the characteristic phonon distribution of the squeezed state allows to clearly distinguish the pairwise created phonons from the thermal background. Therefore we conclude that analogue to cosmological particle creation effects should be observable in already state-of-the-art ion traps [8, 9].

So far we have only treated collisions of 2 ions. However, collisions of the form (15) permit also exact analytical expressions for the Bogoliubov coefficients of higher normal modes. In the limit of slow and moderate colli-

sions they can be approximated by

$$|\beta_\kappa|^2 \propto \exp \left[-2\pi \frac{\sqrt{\omega_{\text{rad}}^2 - \omega_\kappa^2} - \Delta\Omega_{\text{col}} \frac{\omega_\kappa}{\omega_{\text{ax}}^{\text{in}}}}{\omega_{\text{col}}} \right]. \quad (24)$$

Consequently particle creation in the κ -th normal mode becomes only important if

$$\left(\sqrt{\omega_{\text{rad}}^2 - \omega_\kappa^2} - \Delta\Omega_{\text{col}} \frac{\omega_\kappa}{\omega_{\text{ax}}^{\text{in}}} \right) = \mathcal{O}(\omega_{\text{col}}). \quad (25)$$

For an increasing $\Delta\Omega_{\text{col}}$, considerable creation of pairs of phonons occurs therefore firstly in the mode with the highest ω_κ . For large N in a linear chain of ions, this mode is called the zig-zag mode.

B. Expansion model

Another type of axial motion, which corresponds to an expansion of the mutual distance of the ions, is described by the scale function

$$b_{\text{ex}}(t) = \left(1 - \frac{\Delta\Omega_{\text{ex}}^2}{2(\omega_{\text{ax}}^{\text{in}})^2} (\tanh(\omega_{\text{ex}} t) + 1) \right)^{-\frac{1}{3}}. \quad (26)$$

The parameter $\Delta\Omega_{\text{ex}}^2$ describes the induced jump in the normal mode frequency $\Omega_-^2(t)$, whereas ω_{ex} determines how fast the expansion evolves. Inserting $b_{\text{ex}}(t)$ into (14) yields a differential equation that is discussed in [1] as an example for cosmological particle creation. It can be solved in terms of hypergeometric functions whose asymptotic behaviour is known for $t \rightarrow \pm\infty$. The Bogoliubov coefficient reads

$$|\beta_{\text{ex}}^{\text{ex}}|^2 = \frac{\sinh^2 \left(\frac{\pi}{2} \frac{\Omega_{\text{out}} - \Omega_{\text{in}}}{\omega_{\text{ex}}} \right)}{\sinh \left(\pi \frac{\Omega_{\text{in}}}{\omega_{\text{ex}}} \right) \sinh \left(\pi \frac{\Omega_{\text{out}}}{\omega_{\text{ex}}} \right)}, \quad (27)$$

where

$$\Omega_{\text{out}} = \sqrt{\Omega_{\text{in}}^2 + \Delta\Omega_{\text{ex}}^2}. \quad (28)$$

E.g. for very large ω_{ex} , that means for a sudden quench, the Bogoliubov coefficient can be approximated to

$$|\beta_{\text{ex}}^2| \approx \frac{(\Omega_{\text{out}} - \Omega_{\text{in}})^2}{4\Omega_{\text{in}}\Omega_{\text{out}}}, \quad (29)$$

However, in the case of moderate and slow expansions, i.e., $\omega_{\text{ex}} \ll \Omega_{\text{in}}$ the Bogoliubov coefficients become

$$|\beta_{\text{ex}}|^2 \propto e^{-2\pi\Omega_{\text{in}}/\omega_{\text{ex}}}. \quad (30)$$

IV. ION-ION ENTANGLEMENT

After having discussed the excitation process of pairs of phonons in the last section, we now analyze the conditions to reach entanglement between the ions and how robust this entanglement is against thermal disturbances. We discuss exclusively the case of $N = 2$ ions.

We consider a system, that is initially in a thermal state with sufficiently separated ions to consider them initially uncoupled, i.e., $\Omega_-(t_{\text{in}}) = \omega_{\text{rad}}$. In this case the operators $\hat{\delta}y_{\pm}$ and $\hat{\delta}y_{1/2}$ form two equivalent sets of normal modes and their corresponding initial creation and annihilation operators are linked via

$$\hat{a}_+^{\text{in}} = \frac{1}{\sqrt{2}} (\hat{a}_1^{\text{in}} + \hat{a}_2^{\text{in}}) \quad (31)$$

and

$$\hat{a}_-^{\text{in}} = \frac{1}{\sqrt{2}} (\hat{a}_1^{\text{in}} - \hat{a}_2^{\text{in}}) . \quad (32)$$

Next the system becomes squeezed, for example by an ion collision as discussed in section III. Finally, the ions return to their initial positions, such that the coupling vanishes again.

Firstly, we focus on a small squeezing parameter ξ and small thermal excitations on within the radial mode

$$n_{\text{th}} = \langle \hat{n}_1 + \hat{n}_2 \rangle = 2 \langle \hat{n}_1 \rangle = 2 \langle \hat{n}_2 \rangle = 2 \coth \left(\frac{\hbar \omega_{\text{rad}}}{2k_B T} \right) . \quad (33)$$

Here T is the (initial) temperature and k_B is the Boltzmann constant. We do not consider effects of thermal excitations in the axial modes because there is no coupling between axial and radial normal modes, see Eq. (3). The initial density operator can then be written as

$$\begin{aligned} \hat{\rho}_{\text{th}}^{\text{in}} &= (1 - n_{\text{th}}) + |0\rangle_1 |0\rangle_2 \langle 0|_1 \langle 0|_2 \\ &+ \frac{n_{\text{th}}}{2} |1\rangle_1 |0\rangle_2 \langle 1|_1 \langle 0|_2 + \frac{n_{\text{th}}}{2} |0\rangle_1 |1\rangle_2 \langle 0|_1 \langle 1|_2 \\ &+ \mathcal{O}(n_{\text{th}}^2) . \end{aligned} \quad (34)$$

After the squeezing process described by the operator \hat{S}_{ξ} in Eq. (10), the final density operator reads

$$\hat{\rho}_{\text{th}}^{\text{out}} = \hat{S}_{\xi} \hat{\rho}_{\text{th}}^{\text{in}} \hat{S}_{\xi}^{\dagger} . \quad (35)$$

The partially transposed matrix of $\hat{\rho}_{\text{th}}^{\text{out}}$ possesses the eigenvalues $n_{\text{th}} \pm |\xi_-|$ and becomes consequently negative definite for sufficiently large $|\xi_-|$. With the Peres-Horodecki-criterion [10], which is a sufficient separability criterion for Gaussian states [11], it follows, that the ions are entangled if and only if $|\xi_-| > n_{\text{th}}$.

The former result was obtained by assuming small parameters $|\xi_-|$ and n_{th} . However for Gaussian states such as thermal states and squeezed thermal states (which we consider in our scenario), it is also possible to evaluate the Peres-Horodecki-criterion for finite parameters. This was demonstrated in [12, 13] and recently applied to analogue gravity experiments in [14]. In appendix C we adapt the formalism to our system and conclude that an initial thermal state with thermal excitations n_+ and n_- in the $\delta\hat{q}_{\pm}$ normal modes becomes entangled during a squeezing process in the $\delta\hat{q}_-$ mode if and only if the squeezing parameter satisfies $\sqrt{1 + 2n_-} \sqrt{1 + 2n_+} \exp(-|\xi_-|) < 1$. As expected, in the limit of small squeezing parameters and small thermal excitations $2n_+ = 2n_- = n_{\text{th}}$, this results coincides with the former entanglement criterion $|\xi_-| > n_{\text{th}}$.

V. CONCLUSIONS

We considered the radial modes of two or more ions in a trap which we accelerate in the axial direction. The shaping of the axial motion permits us to control the time-dependent coupling between the radial fluctuations and to create a characteristic excitation in these modes. An advantage of this set-up in comparison to previous proposals lies in exploiting axial and radial motion to allow us to derive realistic parameters to enable the detection of phonon pair creation. This will permit to investigate physics and test proposed schemes as well to create entanglement between ions. The process of phonon pair creation has been predicted to emerge in an analogous way in cosmological particle creation or black hole evaporation, where the entanglement between the partners is related to the entropy of the black hole.

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Appendix A: Bogoliubov coefficients

The solutions of the rocking mode differential equation (14) for the collision model (15) are the associated Legendre polynomials $P_\mu^\nu(z)$ with the substitutions

$$\begin{aligned} z &= \tanh(t), \\ \mu &= i \frac{\Omega_{\text{in}}}{\omega}, \\ \nu &= \frac{1}{2} \left(i \sqrt{4\Delta\Omega_{\text{col}}^2/\omega_{\text{col}}^2 - 1} - 1 \right). \end{aligned} \quad (\text{A1})$$

Their asymptotic behavior is

$$P_\mu^\nu(\tanh(t)) \xrightarrow{t \rightarrow \infty} \frac{e^{\mu t}}{\Gamma(1-\mu)} \quad (\text{A2})$$

and

$$\begin{aligned} P_\mu^\nu(\tanh(t)) \xrightarrow{t \rightarrow -\infty} & \frac{\Gamma(-\mu)}{\Gamma(-\mu-\nu)\Gamma(1-\mu+\nu)} e^{\mu t} \\ & - \frac{\sin(\pi\nu)\Gamma(\mu)}{\pi} e^{-\mu t}. \end{aligned} \quad (\text{A3})$$

The Bogoliubov coefficient β_-^{col} is therefore

$$\beta_-^{\text{col}} = -\frac{\sin(\pi\nu)\Gamma(\mu)}{\pi\Gamma(1-\mu)} = \frac{\sin(\pi\nu)}{\sin(\pi\mu)}. \quad (\text{A4})$$

Back substitution yields finally

$$|\beta_-^{\text{col}}|^2 = \left| \frac{\cosh\left(\frac{\pi}{2} \sqrt{\frac{4\Delta\Omega_{\text{col}}^2}{\omega_{\text{col}}^2} - 1}\right)}{\sinh\left(\frac{\pi\Omega_{\text{in}}}{\omega_{\text{col}}}\right)} \right|^2. \quad (\text{A5})$$

Appendix B: WKB-approximation

We derive here the general exponential behavior of the Bogoliubov coefficients for slow and moderate collisions in a normal mode with frequency $\Omega^2(t)$. Moderate means that we stay away from the critical point, i.e.,

$$\Omega^2(t) > 0, \quad (\text{B1})$$

while slow means that

$$\left| \frac{\dot{\Omega}(t)}{\Omega^2(t)} \right| \ll 1. \quad (\text{B2})$$

For a typical collision $\Omega^2(t)$ reaches its minimum when the ions are closest and the scale functions becomes minimal. Without loss of generality this happens at $t = 0$. As shown in [15], under these conditions a WKB-approximation yields the exponential behavior of the Bogoliubov coefficient as

$$|\beta|^2 \propto \exp \left[-4\Im \left\{ \int_0^{t_*} \Omega(t) dt \right\} \right], \quad (\text{B3})$$

where t_* denotes the root of $\Omega(t)$ in the upper complex plane. Phonon creation happens more likely when the exponent is small. This can be achieved by working with low frequencies $\Omega(t)$ and small values for t_* .

Next, we calculate the exponent explicitly for collisions that are well described by a Taylor expansion

$$\Omega^2(t) \approx \Omega_{\text{min}}^2 + \frac{1}{2} K^2 t^2, \quad (\text{B4})$$

in the region $|t| < |t_*|$, where

$$K^2 = \frac{d^2}{dt^2} \Omega^2(t) \Big|_{t=0} \quad (\text{B5})$$

is the curvature. Their complex root is approximated by

$$t_* \approx i\sqrt{2} \frac{\Omega_{\text{min}}}{K}. \quad (\text{B6})$$

Finally, evaluating (B3) leads to

$$|\beta|^2 \propto \exp \left[-\sqrt{2}\pi \frac{\Omega_{\text{min}}^2}{K} \right]. \quad (\text{B7})$$

For the model collision with $b_{\text{col}}(t)$ in (15), this yields the exponential behavior

$$|\beta_-^{\text{col}}|^2 \propto \exp \left[-2\pi \frac{(\Omega_{\text{in}} - \Delta\Omega_{\text{col}})}{\omega_{\text{col}}} \right], \quad (\text{B8})$$

in agreement with (18).

Appendix C: Covariance matrix formalism

To apply the entanglement criteria for Gaussian states developed in [12, 13] to our system we define the phase space vector with respect to the ion coordinates

$$\hat{\mathbf{R}}_{12} = (\delta\hat{y}_1 \ \delta\hat{y}_1 \ \delta\hat{y}_2 \ \delta\hat{y}_2)^T. \quad (\text{C1})$$

The corresponding covariance matrix reads

$$\sigma_{kl} := \frac{1}{2} \left\langle \hat{R}_k \hat{R}_l + \hat{R}_l \hat{R}_k \right\rangle. \quad (\text{C2})$$

We also define the phase space vector with respect to the normal coordinates

$$\hat{\mathbf{R}}_{\pm} = (\delta\hat{y}_+ \ \delta\hat{p}_+ \ \delta\hat{y}_- \ \delta\hat{p}_-)^T = \mathbf{D} \cdot \hat{\mathbf{R}}_{12}, \quad (\text{C3})$$

with the transformation matrix

$$\mathbf{D} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix}. \quad (\text{C4})$$

The covariance matrices corresponding either to $\hat{\mathbf{R}}_{12}$ or to $\hat{\mathbf{R}}_{\pm}$ are linked via

$$\sigma_{12} = \mathbf{D} \cdot \sigma_{\pm} \cdot \mathbf{D}. \quad (\text{C5})$$

We consider an initial thermal covariance matrix

$$\sigma_{\pm}^{\text{in}} = \frac{1}{2} \begin{pmatrix} 1+2n_+ & 0 & 0 & 0 \\ 0 & 1+2n_+ & 0 & 0 \\ 0 & 0 & 1+2n_- & 0 \\ 0 & 0 & 0 & 1+2n_- \end{pmatrix}. \quad (\text{C6})$$

with the thermal occupation numbers

$$n_{\pm} = \coth \left(\frac{\hbar \Omega_{\text{rad}\pm}}{2k_B T} \right). \quad (\text{C7})$$

Its time evolution during a squeezing process is

$$\sigma_{\pm}^{\text{out}} = \mathbf{S}_{\pm} \cdot \sigma_{\pm}^{\text{in}} \cdot \mathbf{S}_{\pm}^T. \quad (\text{C8})$$

where \mathbf{S}_{\pm} is a symplectic matrix containing the Bogoliubov coefficients

$$\mathbf{S}_{\pm} = \begin{pmatrix} \Re\{\alpha_+\} & \Im\{\alpha_+\} & 0 & 0 \\ -\Im\{\alpha_+\} & \Re\{\alpha_+\} & 0 & 0 \\ 0 & 0 & \Re\{\alpha_- - \beta_-\} & \Im\{\alpha_- + \beta_-\} \\ 0 & 0 & -\Im\{\alpha_- - \beta_-\} & \Re\{\alpha_- + \beta_-\} \end{pmatrix}. \quad (\text{C9})$$

Hence we get

$$\sigma_{12}^{\text{out}} = \mathbf{D} \cdot \mathbf{S}_{\pm} \cdot \mathbf{D} \cdot \sigma_{12}^{\text{in}} \cdot \mathbf{D} \cdot \mathbf{S}_{\pm}^T \cdot \mathbf{D}. \quad (\text{C10})$$

As shown in [12], for Gaussian states the Peres-Horodecki criterion can be formulated as a criterion on the two symplectic eigenvalues λ_{\pm} of the partial transposed covariance matrix

$$(\sigma_{12}^{\text{out}})^{PT} = \mathbf{T} \cdot \sigma_{12}^{\text{out}} \cdot \mathbf{T} \quad (\text{C11})$$

with $\mathbf{T} = \text{diag}(1, -1, 1, 1)$. The ions are entangled if one of the symplectic eigenvalues is smaller than $1/2$.

For our system we obtain the symplectic eigenvalues as the two positive eigenvalues of $i\mathbf{J} \cdot (\sigma_{12}^{\text{out}})^{PT}$ to

$$\begin{aligned} \lambda_{\pm}^{PT} &= \frac{1}{2} \sqrt{1+2n_-} \sqrt{1+2n_+} (|\alpha_-| \pm |\beta_-|)^2 \\ &= \frac{1}{2} \sqrt{1+2n_-} \sqrt{1+2n_+} e^{\pm|\xi_-|}, \end{aligned} \quad (\text{C12})$$

where

$$\mathbf{J} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \quad (\text{C13})$$

Therefore the ions are entangled if $\sqrt{1+2n_-} \sqrt{1+2n_+} \exp(-|\xi_-|) < 1$.

Furthermore, in the case of symmetric squeezing, the Entanglement of Formation E_F can be evaluated explicitly [13]. Squeezing is called symmetric when the two blockdiagonal 2×2 matrices of σ_{12}^{out} possess identical determinants, which is here the case. The Entanglement of Formation is then

$$E_F = \begin{cases} f(\lambda_-^{PT}) & \text{wenn } 0 < \lambda_-^{PT} < \frac{1}{2} \\ 0 & \text{wenn } \frac{1}{2} \leq \lambda_-^{PT} \end{cases}, \quad (\text{C14})$$

with the function

$$f(x) = \frac{(\frac{1}{2} + x)^2}{2x} \ln \left(\frac{(\frac{1}{2} + x)^2}{2x} \right) - \frac{(\frac{1}{2} - x)^2}{2x} \ln \left(\frac{(\frac{1}{2} - x)^2}{2x} \right). \quad (\text{C15})$$